

# VECTOR REPRESENTATIONS OF GRAPHS AND DISTINGUISHING QUANTUM PRODUCT STATES WITH ONE-WAY LOCC

DAVID W. KRIBS<sup>1,2</sup>, COMFORT MINTAH<sup>1</sup>, MICHAEL NATHANSON<sup>3</sup>,  
RAJESH PEREIRA<sup>1</sup>

ABSTRACT. Distinguishing sets of quantum states shared by two parties using only local operations and classical communication measurements is a fundamental topic in quantum information. We introduce a graph-theoretic approach, specifically based on the theory of vector representations of graphs, to the core problem of distinguishing product states with one-way LOCC. We establish a number of results that show how distinguishing such states can be framed in terms of properties of the underlying graphs associated with a set of vector product states. We also present a number of illustrative examples.

## 1. INTRODUCTION

In quantum information theory, we frequently attempt to recover classical information that has been encoded into quantum states. If our quantum system consists of multiple physical subsystems, we encounter instances where the information can be recovered with joint measurements on the subsystems but not with local measurements. [5, 8, 16, 18]. This paradigm makes use of local quantum operations and classical communication (LOCC) and includes many topics such as quantum teleportation and data hiding [4, 12, 26]. There is also a growing body of work on the more restricted problem of one-way LOCC, in which parties must perform their measurements in a prescribed order [10, 15, 19–21, 24, 25, 27, 28].

The corresponding linear algebra problem involves attempting to identify an unknown vector  $|\varphi\rangle$  from an orthonormal set of vectors  $\{|\varphi_i\rangle\}$  in a Hilbert space composite system  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ . In the current discussion, we assume that  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are finite-dimensional complex inner product spaces; and the initial measurement is a set

---

2010 *Mathematics Subject Classification.* 47L90, 46B28, 81P15, 81P45, 81R15.

*Key words and phrases.* quantum communication, quantum states, product states, local operations and classical communication, simple graph, graph clique cover, chordal graph.

of rank one operators that sum to the identity of  $\mathcal{H}_A$ . Exploration of connections with linear algebra go back to the origins of quantum information theory, and all of the citations above can be seen in this light. Our recent work to relate one-way LOCC to operator systems and algebras [19–21] is a continuation of this exploration.

One of the most unexpected phenomena in quantum information is that of “nonlocality without entanglement,” originally identified by Bennett et al. [5]. It says that a set of states  $\{|\varphi_i\rangle\}$  in  $\mathcal{H}_A \otimes \mathcal{H}_B$  can fail to be locally distinguishable even if each of them is a product state, i.e. for each  $i$ , we have  $|\varphi_i\rangle = |\varphi_i\rangle_A \otimes |\varphi_i\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$ . This implies that quantum entanglement is not the only way to take advantage of the peculiarities of quantum information. For any set of product states, we can ask whether it exhibits this phenomenon. The local relationships between product states are modeled using the confusability graph (as identified in [11]) which arises in the study of vector representations of graphs [2, 3, 6, 7, 13, 14, 22, 23].

In this paper, we introduce graph-theoretic techniques, and in particular those associated with vector representations of graphs, to the study of LOCC quantum state distinguishability. We specifically focus on the core problem of one-way LOCC distinguishability of product states, identifying and clarifying new structure for the distinguishability of such sets of states based on associated vector graph representations.

This paper is organized as follows. In the next section we present requisite preliminaries from graph theory, along with the vector representations of graphs that arise from sets of quantum product states and an illustrative one-way LOCC example. We then initiate our analysis in the following section, establishing that one-way distinguishability is equivalent to the existence of a graph clique cover with nice properties. Built on this result, in the section that follows we show that it must be possible to distinguish sets of product states with product measurements when either of the parties can go first in the protocol. In the penultimate section we consider one-way LOCC implications when the underlying graphs have extra properties as identified in graph theory, such as chordal or tree structures. The final section includes an extended analysis for the important special case of domino states [5, 9, 29, 30]. Examples are included throughout our presentation, as are discussions on the differences encountered when different parties go first in a one-way LOCC protocol.

## 2. ALICE AND BOB, AND VECTOR REPRESENTATIONS OF GRAPHS

We begin by recalling basic notation and nomenclature from graph theory, drawing on various entrance points into the literature on the subject [2, 3, 6, 7, 13, 14, 22, 23].

Let  $G = (V, E)$  be a *simple graph* with vertex set  $V$  and edge set  $E$ . For  $v, w \in V$ , we write  $v \sim w$  if the edge  $\{v, w\} \in E$ . The *complement* of  $G$  is the graph  $\overline{G} = (V, \overline{E})$ , where the edge set  $\overline{E}$  consists of all two-element sets from  $V$  that are not in  $E$ . Another graph  $G'$  is a *subgraph* of  $G$ , written  $G' \leq G$ , if  $V' \subseteq V$  and  $E' \subseteq E$  with  $v, w \in V'$  whenever  $\{v, w\} \in E'$ .

Some fundamental graphs for fixed  $n \geq 1$  include: the *path graph*  $P_n = (\{v_1, \dots, v_n\}, E)$  such that  $E = \{\{v_i, v_{i+1}\} : 1 \leq i \leq n-1\}$ ; the *cycle graph*  $C_n = (\{v_1, \dots, v_n\}, E)$  such that  $E = \{\{v_i, v_{i+1}\} : 1 \leq i \leq n-1\} \cup \{v_n, v_1\}$ ; and the *complete graph* on  $n$ -vertices (also called a ‘clique’),  $K_n = (\{v_1, \dots, v_n\}, E)$  such that  $E = \{\{v, w\} : v \neq w \in V\}$ .

**Definition 1.** Given a graph  $G = (V, E)$ , a function  $\phi : V \rightarrow \mathbb{C}^d$  is an *orthogonal representation* of  $G$  if for all vertices  $v_i \neq v_j \in V$ ,

$$(1) \quad v_i \not\sim v_j \iff \langle \phi(v_i), \phi(v_j) \rangle = 0.$$

The *minimum vector rank* of  $G$ , denoted  $\text{mvr}(G)$ , is the smallest  $d$  such that  $G$  has an orthogonal representation in  $\mathbb{C}^d$ .

If every vertex of  $G$  is part of at least one edge, then the minimum vector rank coincides with the *minimum semidefinite rank* of the graph [6, 7], and we will be applying the literature of minimum semidefinite rank to prove our results. Note also that these quantities are defined with respect to the underlying field  $\mathbb{C}$ , and there are examples where the minimum rank increases if we restrict ourselves to real vectors. Note the biconditional built into the definition, which is stronger than conditions for graph colouring. This allows us to uniquely define the graph associated with a function  $\phi$ .

We can now introduce a graph perspective to the setting of LOCC through orthogonal representations. In the LOCC state distinguishability context, we typically have two parties, called Alice and Bob, each with a Hilbert space  $\mathcal{H}_A, \mathcal{H}_B$ , and a set of states in  $\mathcal{H}_A \otimes \mathcal{H}_B$  that they wish to distinguish only using local (quantum) measurement operations on their respective systems and classical communication between them. Here we will focus on the important case of one-way LOCC, where Alice and Bob measure in a prescribed order, with one party communicating the results of their measurement to the other, allowing

for the second party to complete the measurement based on that result. We also consider the situation in which the states are all product states; that is, states of the form  $|\psi_A\rangle \otimes |\psi_B\rangle$ .

**Definition 2.** *Given a set of product states  $\{|\psi_k^A\rangle \otimes |\psi_k^B\rangle\}_{k=1}^r$  on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . The graph of these states from Alice's perspective is the unique graph  $G_A$  with vertex set  $V = \{1, 2, \dots, r\}$  such that  $\{|\psi_k^A\rangle\}$  is an orthogonal representation of  $G_A$ . Likewise, the graph of the states from Bob's perspective is the graph  $G_B$  with vertex set  $V$  such that  $\{|\psi_k^B\rangle\}$  is an orthogonal representation of  $G_B$ .*

Note that by definition the obvious maps on  $V_A$  and  $V_B$  generated by this identification yield orthogonal representations of  $G_A$  and  $G_B$ .

Consider the following illustrative example, which we also analyze from the one-way LOCC perspective as a prelude to what follows. We will always use the standard notation  $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$  for a fixed orthonormal basis of  $\mathbb{C}^d$ .

**Example 1.** Consider the following set of five (unnormalized) states in  $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^4 \otimes \mathbb{C}^3$ :

$$\begin{aligned} |\psi_1\rangle &= |0\rangle \otimes (|0\rangle + |2\rangle) \\ |\psi_2\rangle &= (|0\rangle + |1\rangle) \otimes |1\rangle \\ |\psi_3\rangle &= (|1\rangle + |2\rangle) \otimes |2\rangle \\ |\psi_4\rangle &= (|2\rangle + |3\rangle) \otimes (|0\rangle - |1\rangle) \\ |\psi_5\rangle &= |3\rangle \otimes (|0\rangle + |1\rangle + |2\rangle). \end{aligned}$$

Looking at the set of Alice's vectors, we can see it as an orthogonal representation of the graph  $G_A = P_5$  in  $\mathbb{C}^4$ ; with  $|\psi_1^A\rangle = |0\rangle$ ,  $|\psi_2^A\rangle = |0\rangle + |1\rangle$ ,  $|\psi_3^A\rangle = |1\rangle + |2\rangle$ ,  $|\psi_4^A\rangle = |2\rangle + |3\rangle$ ,  $|\psi_5^A\rangle = |3\rangle$ . The set of Bob's vectors is an orthogonal representation of the 'house' graph  $G_B = \overline{P}_5$  in  $\mathbb{C}^3$ .

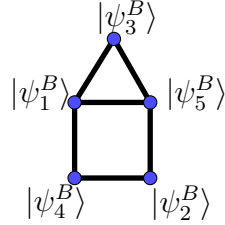


FIGURE 1. Complement of  $P_5$ , represented in  $\mathbb{C}^3$  in this example.

Notice that these states can be distinguished with one-way LOCC with Alice goes first: If Alice measures in the standard basis, then for each of her possible outcomes, there are only two remaining possibilities; and these are orthogonal from Bob's perspective. In general, in order to distinguish our states with one-way LOCC, every measurement outcome of Alice's needs to eliminate possibilities so that the remaining possible states are mutually orthogonal on Bob's side. For instance, in this example if Alice gets a measurement outcome of  $|0\rangle$ , Bob is left to distinguish between the vectors  $\{|0\rangle + |2\rangle, |1\rangle\}$ , which are mutually orthogonal.

### 3. GRAPH CLIQUE COVERS AND ONE-WAY LOCC

We will now attempt to categorize the LOCC distinguishability of states in terms of their corresponding graphs. We begin with the notion of a clique cover.

**Definition 3.** *Given a graph  $G = (V, E)$ . A set of graphs  $\{G_i = (V_i, E_i)\}$  covers  $G$  if  $V = \cup_i V_i$  and  $E = \cup_i E_i$ .*

*A collection of graphs  $\{G_i\}$  is a clique cover for  $G$  if  $\{G_i\}$  covers  $G$  and if each of the  $G_i$  is a complete graph (clique).*

*The clique cover number  $cc(G)$  is the smallest possible number of subgraphs contained in a clique cover of  $G$ .*

A clique cover can be thought of as a collection of (not necessarily disjoint) induced subgraphs of  $G$ , each of which is a complete graph. It is a cover if every edge is contained in at least one of the cliques.

In the example of the previous section, the only clique cover of  $P_5$  is the set of edges; while the complement  $\overline{P_5}$  can be clique covered with a three-cycle and three individual edges, as seen in Figure 2.

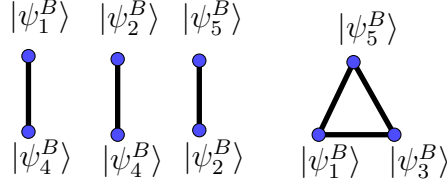


FIGURE 2. Clique cover of  $\overline{P_5}$  with labels as represented in Example 1.

Our first result shows that a set of product states can be perfectly distinguished with one-way LOCC precisely when there is a clique cover with nice properties.

**Theorem 1.** *Given a set of product states in  $\mathcal{H}_A \otimes \mathcal{H}_B$ , let  $G_A$  and  $G_B$  be the graphs of the states from Alice and Bob's perspectives, respectively. Let  $\phi : V_A \rightarrow \mathcal{H}_A$  be the association of vertices with Alice's states and assume that the set  $\{\phi(v) : v \in V\}$  spans  $\mathcal{H}_A$ .*

*Then the states are distinguishable with one-way LOCC with Alice measuring first if and only if there exists*

- (1) *a graph  $G$  satisfying  $G_A \leq G \leq \overline{G_B}$ ,*
- (2) *a clique cover  $\{V_j\}_{j=1}^k$  of  $G$ , and,*
- (3) *a direct sum decomposition  $\mathcal{H}_A = \bigoplus_{j=1}^k \mathcal{S}_j$  with the property that for all  $v \in V_A$ , the support of  $\phi(v)$  is contained in  $\bigoplus_{\{j:v \in V_j\}} \mathcal{S}_j$ .*

*Proof.* One direction of the proof is straightforward: suppose we have a graph  $G$  with  $G_A \leq G \leq G_B$ , a clique cover  $\{V_j\}_{j=1}^k$  of  $G$ , and the decomposition  $\mathcal{H}_A = \bigoplus_{j=1}^k \mathcal{S}_j$  along with the vertex assumption of (3). For each  $j$  define  $Q_j$  as the projection of  $\mathcal{H}_A$  onto  $\mathcal{S}_j$ , so  $\{Q_j\}$  is a (von Neumann) POVM on  $\mathcal{H}_A$ . If Alice gets the outcome  $j$  from the associated measurement, then  $Q_j \phi(v) \neq 0$ , which implies that  $v \in V_j$ . Since the vertices in  $V_j$  form a clique in  $G \leq \overline{G_B}$ , they form a disconnected set in  $G_B$ , reflecting the fact that they are mutually orthogonal. Hence, Bob can distinguish them once he knows Alice's outcome.

The other direction of the proof requires more care. Let  $\{Q_j\}_{j=1}^k$  be a measurement on Alice's system that allows Bob to complete a perfect discrimination of our states. Then for each  $Q_j$ , we define  $V_j = \{v \in V : Q_j|\phi(v)\rangle \neq 0\}$ . It is necessary that the vertices in  $V_j$  form a clique in  $\overline{G}_B$  for Bob to be able to distinguish the remaining possibilities. Define  $G$  to be the union (both vertices and edges) of the cliques induced by the  $V_j$ . By construction, this is a subgraph of  $\overline{G}_B$  and the  $V_j$  form a clique cover of  $G$ .

On the other hand, if  $\langle\phi(u)|\phi(v)\rangle \neq 0$ , then  $\langle\phi(u)|Q_j|\phi(v)\rangle \neq 0$  for some  $j$ , which means that  $u$  and  $v$  are both in  $V_j$ . Hence every edge in Alice's graph  $G_A$  is contained in one of the cliques determined by some  $V_j$ . Thus we see that  $G_A$  is a subgraph of  $G$ , and we get  $G_A \leq G \leq \overline{G}_B$  as desired.

For each measurement operator  $Q_j$ , define  $\mathcal{R}_j$  to be the range of  $Q_j$ . Then we can define  $\mathcal{S}_1 = \mathcal{R}_1$  and for all  $j > 1$ ,

$$\mathcal{S}_j = \mathcal{R}_j \cap \left( \bigcap_{i=1}^{j-1} \mathcal{R}_i^\perp \right).$$

By definition, this means that if  $i < j$  then  $\mathcal{S}_j$  is a subspace of  $\mathcal{R}_i^\perp$  and  $\mathcal{S}_i$  is a subspace of  $\mathcal{R}_i$ , and so the subspaces are mutually orthogonal. Also, by construction, each  $\mathcal{R}_j \subseteq \bigoplus_{i=1}^j \mathcal{S}_i$ . Hence  $\mathcal{R}_j \subseteq \bigoplus_{i=1}^k \mathcal{S}_i$  for every  $j$ . Since  $\sum_j Q_j = I$ , the linear span of the  $\{\mathcal{R}_j\}$  is all of  $\mathcal{H}_A$ , which implies that

$$\mathcal{H}_A = \bigoplus_{j=1}^k \mathcal{S}_j.$$

Finally, we see that for each vertex  $v$ , if  $\phi(v)$  has a nontrivial component in  $\mathcal{S}_j$  then it has a component in  $\mathcal{R}_j$ , implying that  $v \in V_j$ . Thus, the support of  $\phi(v)$  is contained in  $\bigoplus_{j:v \in V_j} \mathcal{S}_j$ .  $\square$

**Example 2.** Returning to our example from the previous section in light of the theorem, the graph  $G$  is the path  $P_5$ , and  $G_A = P_5 = \overline{G}_B$ . The clique cover is simply the collection of edges, with corresponding subspaces of  $\mathcal{H}_A$  as in the theorem given by:

$$\begin{aligned} V_0 &= \{v_1, v_2\} & \mathcal{S}_0 &= \{(|1\rangle + |2\rangle), (|2\rangle + |3\rangle), |3\rangle\}^\perp = \text{span}\{|0\rangle\} \\ V_1 &= \{v_2, v_3\} & \mathcal{S}_1 &= \{|0\rangle, (|2\rangle + |3\rangle), |3\rangle\}^\perp = \text{span}\{|1\rangle\} \\ V_2 &= \{v_3, v_4\} & \mathcal{S}_2 &= \{|0\rangle, (|0\rangle + |1\rangle), |3\rangle\}^\perp = \text{span}\{|2\rangle\} \\ V_3 &= \{v_4, v_5\} & \mathcal{S}_3 &= \{|0\rangle, (|0\rangle + |1\rangle), (|1\rangle + |2\rangle)\}^\perp = \text{span}\{|3\rangle\}. \end{aligned}$$

This gives us an orthonormal basis in which to measure in, as previously noted. Note that if instead we have Bob measure first, we can see right

away that this is not possible. The graph corresponding to Bob's states is the house graph; and a minimum clique cover contains four cliques. Since we are in  $\mathbb{C}^3$ , there is no clique cover of size less than or equal to  $d$  and it is not possible to meet the conditions of the theorem. In conclusion, if we are to distinguish the states in Example 1 using one-way LOCC, it must be with Alice measuring first.

Let us point out a consequence of the construction in the proof of the theorem.

**Corollary 1.** *If Alice and Bob have a set of product states that can be distinguished with one-way LOCC with Alice going first, then it is possible to distinguish them with a von Neumann measurement; in particular, it is not necessary to use a general POVM.*

As a consequence of Theorem 1, we thus have graph-theoretical necessary conditions for one-way LOCC with product states. As a one-way LOCC measurement by Alice corresponds to a clique cover of a subgraph  $\overline{G}_B$ , this creates interest in its clique cover number.

**Corollary 2.** *If a set of product states in  $\mathcal{H}_A \otimes \mathcal{H}_B$  are distinguishable with one-way LOCC with Alice measuring first, then*

$$\text{cc}(\overline{G}_B) \leq \dim(\mathcal{H}_A)$$

*Proof.* If we let  $d_A = \dim(\mathcal{H}_A)$ , then Theorem 1 states that if Alice can initiate a one-way LOCC measurement, then there exists graph  $G$  and a clique cover of size  $k$  that corresponds to a decomposition of  $\mathcal{H}_A$ . This means that  $k$  is at most the dimension of the space. Since  $\text{cc}(G)$  is the size of the smallest clique cover of  $G$ , we have

$$\text{cc}(G) \leq k \leq \dim(\mathcal{H}_A).$$

We also know that  $G$  is a subgraph of  $\overline{G}_B$ , and they share the same vertex set. Hence,  $\text{cc}(\overline{G}_B) \leq \text{cc}(G)$  and the corollary follows.  $\square$

We also have the following consequence.

**Corollary 3.** *Suppose that Alice and Bob have a set of product states that can be distinguished with one-way LOCC with Alice going first, and let  $G$  be the graph defined in Theorem 1. If  $d = \dim \mathcal{H}_A$ , then there exists a graph  $G'$  on  $(n + d)$  vertices such that  $G$  is an induced subgraph of  $G'$ ; the additional  $d$  vertices form an independent set; and*

$$\text{mvr}(G) \leq \text{cc}(G) \leq \text{cc}(G') = \text{mvr}(G') = d$$

*Proof.* Let  $\{|\phi_i\rangle\langle\phi_i|\}$  be an orthonormal measurement on Alice's side that starts the LOCC process. Construct the graph  $G'$  by adding a vertex for each of the  $|\phi_i\rangle$ . These vertices form an independent set since



they are mutually orthogonal. Each of these new vertices is simplicial, in that its neighbors form a complete graph.

We can extend the clique cover of  $G$  to a clique cover of  $G'$ . The size of the cover will increase if there is a gap between  $\text{cc}(G)$  and  $d$ .

Since our  $d$  new vertices form an independent set,  $\text{mvr}(G') \geq d$ ; and since the new clique cover is of size  $d$ ,  $\text{cc}(G') \leq d$ . Since  $\text{mvr}(G) \leq \text{cc}(G)$  for any graph, the conclusion follows.  $\square$

#### 4. PRODUCT MEASUREMENTS AND ONE-WAY LOCC

We can build on Theorem 1 to derive the following important and intuitive consequence, in the case that order of the parties does not matter in a one-way LOCC protocol with an extra condition on the states.

**Theorem 2.** *Suppose we have a set of product states  $\{|\psi_i^A\rangle \otimes |\psi_i^B\rangle\}$  in  $\mathcal{H}_A \otimes \mathcal{H}_B$ , and let  $G_A$  and  $G_B$  be the graphs for Alice and Bob. Suppose further that  $G_A = \overline{G_B}$ ; that is, for every  $i \neq j$ ,*

$$\langle \psi_i^A | \psi_j^A \rangle = 0 \iff \langle \psi_i^B | \psi_j^B \rangle \neq 0.$$

*If it is possible to distinguish the states with one-way LOCC with Alice going first as well as with one-way LOCC with Bob going first, then it is possible to distinguish them with a product measurement.*

*Proof.* From two applications of Theorem 1, we have that  $\mathcal{H}_A = \bigoplus_j \mathcal{S}_j$  and similarly,  $\mathcal{H}_B = \bigoplus_j \mathcal{O}_j$ . This implies that the entire Hilbert space has a product decomposition  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \bigoplus_{i,j} (\mathcal{S}_j \otimes \mathcal{O}_i)$ , with the corresponding graph implications of the result.

Suppose that Alice and Bob each perform their one-way measurements and receive the outcomes  $\mathcal{S}_j$  and  $\mathcal{O}_i$ . This means that  $v \in V_j \cap W_i$ , where  $V_j$  induces a clique in  $\overline{G_B}$  and  $W_i$  induces a clique in  $\overline{G_A}$ . Suppose that two vertices are each contained in  $V_j \cap W_i$ . Then they are connected by an edge in both  $\overline{G_B}$  and  $\overline{G_A} = G_B$ , which is a contradiction. Hence for each pair  $i, j$ , we have  $|V_j \cap W_i| \leq 1$ , which means that Alice and Bob can determine the identity of their state without further communication.  $\square$

The example below shows that the assumption  $\overline{G_A} = G_B$  is necessary in the statement of Theorem 2. If  $\overline{G_A} \neq G_B$ , then Alice and Bob have extra flexibility, which weakens the assumption of one-way LOCC in both directions.

**Example 3.** Consider the following states in  $\mathbb{C}^3 \otimes \mathbb{C}^3$ :

$$\begin{aligned} |\psi_1\rangle &= |0\rangle \otimes |0\rangle \\ |\psi_2\rangle &= (|1\rangle + |2\rangle) \otimes |0\rangle \\ |\psi_3\rangle &= (|1\rangle - |2\rangle) \otimes |0\rangle \\ |\psi_4\rangle &= |0\rangle \otimes (|1\rangle + |2\rangle) \\ |\psi_5\rangle &= |0\rangle \otimes (|1\rangle - |2\rangle) \\ |\psi_6\rangle &= |1\rangle \otimes |1\rangle \\ |\psi_7\rangle &= |2\rangle \otimes |2\rangle. \end{aligned}$$

Observe in this case that  $G_A = G_B = C_3 \cup C_4$  is the disjoint union of two cycle graphs, with respective labelling determined by the ordered sets:

$$\{|\psi_1^A\rangle, |\psi_4^A\rangle, |\psi_5^A\rangle\}, \quad \{|\psi_2^A\rangle, |\psi_6^A\rangle, |\psi_3^A\rangle, |\psi_7^A\rangle\},$$

and

$$\{|\psi_1^B\rangle, |\psi_2^B\rangle, |\psi_3^B\rangle\}, \quad \{|\psi_4^B\rangle, |\psi_6^B\rangle, |\psi_5^B\rangle, |\psi_7^B\rangle\}.$$

On the other hand,  $\overline{G_A}$  is a graph not isomorphic to this graph, given by the disjoint union of three isolated vertices ( $\{1\}, \{4\}, \{5\}$ ) and two isolated edges ( $\{2, 3\}, \{6, 7\}$ ).

These states are distinguishable with one-way LOCC: Alice measures in the basis  $\{|0\rangle, (|1\rangle \pm |2\rangle)\}$ . If she gets the outcome  $|0\rangle$ , then Bob completes the measurement using the same measurement that Alice did; but if she gets either of the remaining outcomes, Bob should measure in the standard basis. It is clear that these states are symmetric with respect to Alice and Bob, so they can also be distinguished with one-way LOCC with Bob going first and corresponding measurement adjustments.

**Proposition 1.** *The 7 states described in Example 3 cannot be distinguished with a product measurement, even though they can be distinguished with one-way LOCC in both directions.*

*Proof.* This fact follows from the observation that Alice's initial measurement for one-way LOCC is unique. Her measurement operators must eliminate at least four possibilities so that Bob has only three orthogonal possibilities remaining. This means there are at most  $\binom{7}{4}$  possible measurement operators, but it must also be true that the span of these four vectors cannot be all of  $\mathbb{C}^3$ , since they are in the kernel of a nonzero operator. There are only 5 sets of 4 of the  $|\psi_i\rangle$  that do not span all of  $\mathbb{C}^3$ . It must also be true that the complementary sets of 3 vectors must be mutually orthogonal on Bob's side. This eliminates 2

of the sets, leaving us with the following three sets of vectors:

$$\{|\psi_4\rangle, |\psi_5\rangle, |\psi_6\rangle, |\psi_7\rangle\}, \quad \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_5\rangle\}, \quad \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\}.$$

These correspond to the measurement  $\{|0\rangle, (|1\rangle + |2\rangle), (|1\rangle - |2\rangle)\}$  described above. There are no other options for Alice's initial measurement or for Bob's. As we noted, the second measurement is dependent on the outcome of the first; there is no way to accomplish this without knowing the other outcome.  $\square$

In table form, we can see the possibilities with this example. The rows and columns represent Alice and Bob's measurements, respectively. The table entries represent the identity of  $|\psi_i\rangle$ :

	$ 0\rangle$	$ 1\rangle +  2\rangle$	$ 1\rangle -  2\rangle$	
$ 0\rangle$	1	4	5	$\{1, 4, 5\}$
$ 1\rangle +  2\rangle$	2	6, 7	6, 7	$\{2, 6, 7\}$
$ 1\rangle -  2\rangle$	3	6, 7	6, 7	$\{3, 6, 7\}$
	$\{1, 2, 3\}$	$\{4, 6, 7\}$	$\{5, 6, 7\}$	

This implies that Theorem 2 is not true without the assumption that  $G_A = \overline{G_B}$ . As noted above, for this example  $G_A = G_B = C_3 \cup C_4$ , the union of a triangle with a four-cycle. This has a clique cover number of 5. However, if you add a single diagonal to the four-cycle, you get an intermediate graph which is a subgraph of  $\overline{G_B}$  and which has clique cover number 3, and it is this measurement that corresponds to the one-way LOCC measurement. Crucially, this diagonal edge is the same from both Alice's and Bob's perspective (corresponding in both cases to states  $|\psi_6\rangle$  and  $|\psi_7\rangle$ ); it must be part of the initial clique cover, which is why the product measurement does not work.

## 5. CHORDAL GRAPHS AND ONE-WAY LOCC

We next conduct further one-way LOCC analysis on some special cases considered in graph theory.

**Definition 4.** *A graph  $G$  is chordal if it contains no induced cycles  $C_n$  of size greater than three.*

*A vertex  $v$  in a graph  $G$  is simplicial if its connected neighbourhood forms a clique.*

We note that the class of chordal graphs includes trees. Also, in connected chordal graphs, the clique cover number is equal to the minimum vector rank [17].

**Proposition 2.** *Suppose that Alice and Bob have a set of mutually orthogonal product states, and let  $G_B$  be the graph corresponding to*

*Bob's side.* If the graph  $G = \overline{G}_B$  is chordal, then the states can be distinguished with one-way LOCC with Alice going first.

*Proof.* This proof is inspired by the construction of an OS-set for chordal graphs in Hackney, et al. [17] and uses the fact that every chordal graph contains at least one simplicial vertex. We build a direct sum decomposition corresponding to Alice's measurement as in the proof of Theorem 1.

Initialize  $j = 1$  and  $V_1 = V$ . The algorithm is as follows:

- Let  $G_j$  be the induced subgraph of  $G$  on vertices  $V_j$ .
- $G_j$  is an induced subgraph of a chordal graph, so it is chordal, implying that it has a simplicial vertex. Let  $v_j$  be a simplicial vertex of  $G_j$ .
- Define  $\mathcal{K}_j = \text{span}\{\phi(v) : v \in V_j, v \not\sim v_j\}$  to be the span of the nonneighbours of  $v_j$  in  $G_j$ ; and set  $\mathcal{R}_j = \mathcal{K}_j^\perp$ .
- Set  $\mathcal{S}_j = \mathcal{R}_j \cap \left(\bigcap_{i=1}^{j-1} \mathcal{R}_i^\perp\right)$  as in the proof above. Since  $\phi(v_j) \in \mathcal{R}_j$  and  $\phi(v_j)$  has a nonzero component in  $(\bigoplus_{i=1}^{j-1} \mathcal{S}_i)^\perp$  by assumption (below), this space is non-trivial.
- Let  $V_{j+1}$  be the set of vertices  $v \in V$  such that  $\phi(v)$  has a nonzero component in  $(\bigoplus_{i=1}^j \mathcal{S}_i)^\perp$ .
- If  $V_{j+1} \neq \emptyset$ , increase  $j$  and iterate. Otherwise stop.

When the process terminates,  $V_{j+1} = \emptyset$  and  $\phi(v) \in \bigoplus_{i=1}^j \mathcal{S}_i$  for all  $v \in V$ . Hence,

$$\bigoplus_{i=1}^j \mathcal{S}_i = \text{span}\{\phi(v) : v \in V_j, v \not\sim v_j\} = \mathcal{H}_A.$$

If two vertices  $u$  and  $v$  have  $\phi(u), \phi(v)$  each with nonzero components in  $\mathcal{S}_i$ , then both are neighbours of  $v_i$  in  $G_i$ . Since  $v_i$  is simplicial in  $G_i$ ,  $u \sim v$  in  $G_i$ , implying that  $u \sim v$  in  $G$ .

This implies that we have constructed a measurement for Alice that will enable Bob to distinguish his states.  $\square$

Note that the proof of the proposition only uses the fact that every chordal graph has a simplicial vertex and that every induced subgraph of a chordal graph is chordal. Also observe that the result applies to the case that  $\overline{G}_B$  is a tree. A similar argument can be used to extend this result to graphs that are  $k$ -trees considered in [1].

**Definition 5.** For  $k \geq 1$ , we can define the class of  $k$ -trees:  $G$  is a  $k$ -tree if  $G = K_{k+1}$ .  $G$  is also a  $k$ -tree if there exists a vertex  $v$  such that (a) the neighbours of  $v$  form a complete graph  $G = K_k$  and (b) the graph  $G' = G - v$  formed by deleting  $v$  is a  $k$ -tree.  $G$  is a partial  $k$ -tree if it is a subgraph of a  $k$ -tree.

This inductive definition allows us to build up a large class of  $k$ -trees and partial  $k$ -trees.

**Proposition 3.** *Suppose that Alice and Bob have a set of mutually orthogonal product states, and let  $G_B$  be the graph corresponding to Bob's side. If there exists a graph  $G$  such that  $G_A \leq G \leq \overline{G_B}$  and  $G$  is a  $k$ -tree, then the states can be distinguished with one-way LOCC with Alice going first.*

*Proof.* We prove this by induction on the number of vertices in  $G$ . If  $G = k + 1$ , then  $G = K_{k+1}$  and Alice's measurement is irrelevant; Bob can distinguish the states by himself.

Now suppose that the result is true for all  $k$ -trees with fewer than  $n$  vertices and let  $G$  be a  $k$ -tree on  $n$  vertices. Let  $v_n$  be the vertex added in the last step of the construction of the  $k$ -tree. Let  $|\phi\rangle = \phi(v_n)$  and let Alice measure with  $\{|\phi\rangle\langle\phi|, I - |\phi\rangle\langle\phi|\}$ . If Alice gets the first outcome, then she knows that the state is in the closed neighbourhood of  $v_n$ . These are states that can be distinguished by Bob.

If Alice gets the second outcome, her state is now in the state  $\phi'(v) = (I - |\phi\rangle\langle\phi|)\phi(v)$  for some  $v$ . Let  $G' = G - v$  be the induced subgraph of  $G$  formed by deleting  $v$ . Let  $u$  be any neighbour of  $v_n$ . Then  $u$  has at least  $k$  other neighbours in  $G$ , which means that there exists a vertex in  $V$  that is adjacent to  $u$  but not  $v_n$ . This implies that  $\phi(u) \neq \phi(v)$ , which implies that  $\phi'(u) \neq 0$ . Hence  $\phi'$  is an orthogonal representation of  $G'$ . But  $G'$  is a  $k$ -tree by definition, and so by our inductive assumption, Alice can complete her measurement to put Bob in a position to determine their state.  $\square$

## 6. THE GRAPH OF A DOMINO STATE DIAGRAM

The original domino states were an orthonormal basis of  $\mathbb{C}^3 \otimes \mathbb{C}^3$  consisting entirely of product states. They were constructed in [5] as an example of a set of orthogonal product states that cannot be distinguished by LOCC. A domino diagram was constructed in [5] to help readers better picture this construction. This has motivated generalizations of domino states in larger Hilbert spaces; examples of these can be found in [9, 29, 30].

We will describe sets of generalized domino states on  $\mathbb{C}^m \otimes \mathbb{C}^n$  in terms of the associated domino diagram. We define a domino diagram to be a partition of the  $m \times n$  rectangular chessboard into a set of generalized dominos. The generalized dominos each have positive integer length and width one and are placed either horizontally or vertically on the chessboard aligned to the  $m \times n$  grid. We assume that the

chessboard is on a torus, so that dominos which exit off an edge simply continue on the other side.

Label the rows on the chessboard  $0, 1, 2, \dots, m-1$  from top to bottom and the columns  $0, 1, 2, \dots, n-1$ . This identifies each square on the board with an element of  $\mathbb{Z}_m \times \mathbb{Z}_n$ . We can then construct a bijection from  $\mathbb{Z}_m \times \mathbb{Z}_n$  to a set of generalized domino states in  $\mathbb{C}^m \otimes \mathbb{C}^n$  as follows: If there is a horizontal domino on row  $r$  whose endpoints are the  $(r, b+1)$  and  $(r, b+s)$  squares, then the point  $(r, b+j)$  with  $1 \leq j \leq s$  gets mapped to  $\sum_{k=1}^s \alpha_r^k \omega^{jk} |r\rangle |b+k\rangle$  where  $\omega$  is the primitive  $s$ th root of unity and  $\alpha_r$  is an arbitrary phase associated with row  $r$ . If there is a vertical domino on column  $c$  whose endpoints are the  $(b+1, c)$  and  $(b+s, c)$  squares, then the  $(b+j, c)$  square with  $1 \leq j \leq s$  gets mapped to  $\sum_{k=1}^s \beta_c^k \omega^{jk} |b+k\rangle |c\rangle$  where  $\omega$  is again the primitive  $s$ th root of unity and now  $\beta_c$  is an arbitrary phase associated with column  $c$ .

We can obtain the graphs  $G_A$  and  $G_B$  of a set of generalized domino states directly from its associated domino diagram.

**Definition 6.** *Let  $\mathcal{D}$  be an  $m \times n$  domino diagram. Then the row (respectively column) graph of  $\mathcal{D}$  is the graph whose vertex set is the set of  $mn$  unit squares on the rectangular chessboard. Two distinct squares are adjacent if and only if one or both of the following two conditions are satisfied:*

- (1) *The two squares lie in the same row (respectively column) on the chessboard.*
- (2) *The two squares each lie in two different dominoes  $D_1$  and  $D_2$  and there exists at least one row (respectively column) of  $\mathcal{D}$  which intersects both  $D_1$  and  $D_2$ .*

It is an easy exercise to show that the row and column graphs of a domino diagram are the graphs  $G_A$  and  $G_B$  of the set of generalized domino states corresponding to the domino diagram. Since  $G_A$  and  $G_B$  can never have an edge in common,  $G_A \leq \overline{G_B}$  and  $G_B \leq \overline{G_A}$ . We get equality ( $G_A = \overline{G_B}$  and  $G_B = \overline{G_A}$ ) if and only if any two dominoes in the diagram have either a common row or a common column that intersects them.

The row and column graphs of domino diagrams have a nice structure that allows us to bound the clique cover number from below:

**Proposition 4.** *Let  $G_A$  and  $G_B$  be the row and column graphs of an  $m \times n$  domino diagram. Then  $\text{cc}(\overline{G_A}) \geq n$  and  $\text{cc}(\overline{G_B}) \geq m$ .*

*If in addition we know that  $G_A = \overline{G_B}$ , then  $\text{cc}(G_B) \geq m - v + v^2$  and  $\text{cc}(\overline{G_A}) \geq n - h + h^2$ , where  $v$  and  $h$  are the lengths of the longest vertical and horizontal dominoes in the diagram.*

*Proof.* Without loss of generality, assume that the largest horizontal domino lies in row 0, and recall the association of domino states with elements of  $\mathbb{Z}_m \times \mathbb{Z}_n$ . The  $n$  states of the form  $\{(0, j)\}$  lies in the first row and thus form a clique in  $G_A$  and an independent set in  $\overline{G_A}$ . Since the clique cover number of a graph is at least its independence number, we have  $\text{cc}(\overline{G_A}) \geq \text{cc}(H) \geq n$ . This proves the first bound.

For the second bound, let  $H$  be the induced subgraph of  $\overline{G_A}$  on the  $2n$  vertices  $\{(i, j) : i \in \{0, 1\}\}$ . Because the set of states associated with each row form an independent set in  $\overline{G_A}$ , the graph  $H$  is bipartite. This implies that the clique cover number of  $H$  is simply the number of edges in  $H$ , which is the sum of the degrees of the vertices in a single partition.

If we assume that  $G_B = \overline{G_A}$ , then each vertex  $(0, j)$  is adjacent to  $(1, j)$  in  $H$ , since they are in the same column. This implies that the degree of vertex  $(0, j)$  is at least one. If there is a horizontal domino of length  $h$  in row zero, then the degree of each of those corresponding vertices is at least  $h$ , since they are connected to each of the columns they appear in. This implies that if  $G_B = \overline{G_A}$ ,

$$\text{cc}(\overline{G_A}) \geq \text{cc}(H) = \sum_j \deg(0, j) \geq h(h) + (n - h)(1) = n - h + h^2.$$

The proofs bounding  $\text{cc}(\overline{G_B})$  are similar, and the result follows.  $\square$

We can obtain the following one-way LOCC consequence based on this graph analysis and our earlier results.

**Corollary 4.** *Given a domino diagram covering an  $m \times n$  grid, and consider the basis of  $\mathbb{C}^m \otimes \mathbb{C}^n$  associated with the diagram. Suppose further that the corresponding graphs have  $G_A = \overline{G_B}$ .*

*If the grid contains a vertical tile of length at least 2, then the basis cannot be distinguished with one-way LOCC with Alice going first; and if the grid contains a horizontal tile of length at least 2, then the basis cannot be distinguished with one-way LOCC with Bob going first.*

*Proof.* This follows from the fact that if  $v > 1$ , then  $\text{cc}(G_B) = \text{cc}(\overline{G_A}) \geq n - h + h^2 > n$ . We can then apply the result above together with Corollary 2 to show that one-way distinguishability is not possible. A similar argument can be used when  $h > 1$ .  $\square$

**Remark 1.** This observation can be used to show that the states described by many of the domino diagrams in the literature cannot be distinguished by one way LOCC. For instance, the original  $3 \times 3$  domino diagram from [5] has two distinct vertical dominoes of length two that intersect in a row; hence  $\text{cc}(G_A) \geq 4$ . This diagram also has

two distinct horizontal dominoes that intersect in one column; hence  $\text{cc}(G_B) \geq 4$ . Since  $G_A = \overline{G_B}$  and  $G_B = \overline{G_A}$ , these states cannot be distinguished by one-way LOCC by either Alice or Bob going first by Corollary 2 as well.

*Acknowledgements.* D.W.K. was partly supported by NSERC and a University Research Chair at Guelph. C.M. was partly supported by Mitacs and the African Institute for Mathematical Sciences. R.P. was partly supported by NSERC. M.N. acknowledges the ongoing support of the Saint Mary's College Office of Faculty Development.

#### REFERENCES

- [1] Fatemeh Alinaghipour, Shaun Fallat, and Karen Meagher. On the relationship between zero forcing number and certain graph coverings. *Special Matrices*, 2:30–45, 2014.
- [2] Francesco Barioli, Wayne Barrett, Shaun M Fallat, Tracy H Hall, Leslie Hogben, Bryan Shader, Pauline Van Den Driessche, and Hein Van Der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433(2):401–411, 2010.
- [3] Francesco Barioli, Shaun M Fallat, Lon H Mitchell, and Sivaram K Narayan. Minimum semidefinite rank of outerplanar graphs and the tree cover number. *Electronic Journal of Linear Algebra*, 22(2):10–21, 2011.
- [4] Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70(13):1895, 1993.
- [5] Charles H Bennett, David P DiVincenzo, Christopher A Fuchs, Tal Mor, Eric Rains, Peter W Shor, John A Smolin, and William K Wootters. Quantum nonlocality without entanglement. *Physical Review A*, 59(2):1070, 1999.
- [6] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R Johnson, Margaret Lay, Terry D Lenker, Lon H Mitchell, Sivaram K Narayan, Amanda Pascoe, and Brian D Sutton. On the minimum semidefinite rank of a simple graph. *Linear and Multilinear Algebra*, 59(5):483–506, 2011.
- [7] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R Johnson, Margaret Lay, Lon H Mitchell, Sivaram K Narayan, Amanda Pascoe, Kelly Steinmetz, Brian D Sutton, et al. On the minimum rank among positive semidefinite matrices with a given graph. *SIAM Journal on Matrix Analysis and Applications*, 30(2):731–740, 2008.
- [8] Anthony Chefles. Condition for unambiguous state discrimination using local operations and classical communication. *Physical Review A*, 69(5):050307, 2004.
- [9] Scott M Cohen. General approach to quantum channel impossibility by local operations and classical communication. *Physical Review Letters*, 118(2):020501, 2017.



- [10] Alessandro Cosentino and Vincent Russo. Small sets of locally indistinguishable orthogonal maximally entangled states. *Quantum Information & Computation*, 14:1098–1106, 2014.
- [11] Runyao Duan, Simone Severini, and Andreas Winter. Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number. *IEEE Transactions on Information Theory*, 59(2):1164–1174, 2013.
- [12] Tilo Eggeling and Reinhard F Werner. Hiding classical data in multipartite quantum states. *Physical Review Letters*, 89(9):097905, 2002.
- [13] Shaun Fallat and Leslie Hogben. Variants on the minimum rank problem: A survey ii. *arXiv preprint arXiv:1102.5142*, 2011.
- [14] Shaun M Fallat and Leslie Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear Algebra and its Applications*, 426(2-3):558–582, 2007.
- [15] Heng Fan. Distinguishability and indistinguishability by local operations and classical communication. *Physical Review Letters*, 92(17):177905, 2004.
- [16] Sibasish Ghosh, Guruprasad Kar, Anirban Roy, and Debasis Sarkar. Distinguishability of maximally entangled states. *Physical Review A*, 70(2):022304, 2004.
- [17] Philip Hackney, Benjamin Harris, Margaret Lay, Lon H Mitchell, Sivaram K Narayan, and Amanda Pascoe. Linearly independent vertices and minimum semidefinite rank. *Linear Algebra and its Applications*, 431(8):1105–1115, 2009.
- [18] Michał Horodecki, Aditi Sen, Ujjwal Sen, Karol Horodecki, et al. Local indistinguishability: More nonlocality with less entanglement. *Physical Review Letters*, 90(4):047902, 2003.
- [19] David W Kribs, Comfort Mintah, Michael Nathanson, and Rajesh Pereira. Operator structures and quantum one-way LOCC conditions. *Journal of Mathematical Physics*, 58(9):092201, 2017.
- [20] David W Kribs, Comfort Mintah, Michael Nathanson, and Rajesh Pereira. One-way LOCC indistinguishable lattice states via operator structures. *Submitted for publication*, 2019.
- [21] David W Kribs, Comfort Mintah, Michael Nathanson, and Rajesh Pereira. Quantum error correction and one-way LOCC state distinguishability. *Journal of Mathematical Physics*, 60(3):032202, 2019.
- [22] László Lovász, Michael Saks, and Alexander Schrijver. Orthogonal representations and connectivity of graphs. *Linear Algebra and its Applications*, 114:439–454, 1989.
- [23] László Lovász, Michael Saks, and Alexander Schrijver. A correction: orthogonal representations and connectivity of graphs. *Linear Algebra and its Applications*, 313(1-3):101–105, 2000.
- [24] Michael Nathanson. Distinguishing bipartite orthogonal states using LOCC: Best and worst cases. *Journal of Mathematical Physics*, 46(6):062103, 2005.
- [25] Michael Nathanson. Three maximally entangled states can require two-way local operations and classical communication for local discrimination. *Physical Review A*, 88(6):062316, 2013.
- [26] Barbara M Terhal, David P DiVincenzo, and Debbie W Leung. Hiding bits in Bell states. *Physical Review Letters*, 86(25):5807, 2001.

- [27] Jonathan Walgate, Anthony J Short, Lucien Hardy, and Vlatko Vedral. Local distinguishability of multipartite orthogonal quantum states. *Physical Review Letters*, 85(23):4972, 2000.
- [28] Nengkun Yu, Runyao Duan, and Mingsheng Ying. Four locally indistinguishable ququad-ququad orthogonal maximally entangled states. *Physical Review Letters*, 109(2):020506, 2012.
- [29] Zhi-Chao Zhang, Fei Gao, Guo-Jing Tian, Tian-Qing Cao, and Qiao-Yan Wen. Nonlocality of orthogonal product basis quantum states. *Physical Review A*, 90(2):022313, 2014.
- [30] Huijuan Zuo, Shuxia Liu, and Yinghui Yang. New constructions of orthogonal product basis quantum states. *International Journal of Theoretical Physics*, 57(5):1597–1603, 2018.

<sup>1</sup>DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF GUELPH, GUELPH, ON, CANADA N1G 2W1

<sup>2</sup>INSTITUTE FOR QUANTUM COMPUTING AND DEPARTMENT OF PHYSICS & ASTRONOMY, UNIVERSITY OF WATERLOO, WATERLOO, ON, CANADA N2L 3G1

<sup>3</sup>DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SAINT MARY'S COLLEGE OF CALIFORNIA, MORAGA, CA, USA 94556